8.3 - Nonhomogeneous Linear Systems

$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$ is a nonhomogeneous system if $\mathbf{F}(t) \neq \mathbf{0}$.

<u>Example:</u> Use the method of undetermined coefficients to solve the given nonhomogeneous system.

$$\mathbf{X}' = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 4t + 9e^{6t} \\ -t + e^{6t} \end{pmatrix}$$

Variation of parameters

The complementary solution is $\mathbf{X}_c(t) = \mathbf{\Phi}(t)\mathbf{C}$.

Definition: $\Phi(t)$ is the **fundamental matrix** of the system.

For a system with independent solutions, the Wronskian is nonzero, so $\mathbf{\Phi}^{-1}$ exists. Since Φ is a solution to $\mathbf{X}' = \mathbf{A}\mathbf{X}$, we have $\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}$. Suppose $\mathbf{X}_p = \mathbf{\Phi}(t)\mathbf{U}(t)$, where $\mathbf{U}(t)$ is a column matrix of unknown functions.

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt$$

Example: Use variation of parameters to solve the given nonhomogeneous system.

$$\frac{dx}{dt} = 2x - y$$
$$\frac{dy}{dt} = 3x - 2y + 4t$$

Example: Use variation of parameters to solve the given nonhomogeneous system.

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} e^{2t}$$

For an initial-value problem of the form $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t),\ X(t_0) = \mathbf{X}_0$, we have $\mathbf{X} = \mathbf{\Phi}\mathbf{C} + \mathbf{\Phi}\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)\,ds \implies \mathbf{X}(t_0) = \mathbf{\Phi}(t_0)\mathbf{C} \implies \mathbf{C} = \mathbf{\Phi}^{-1}(t_0)\mathbf{X}(t_0)$. Thus,

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{X}(t_0) + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)\,ds$$

<u>Example:</u> Solve the given initial-value problem.

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1/t \\ 1/t \end{pmatrix}, \quad \mathbf{X}(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$